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ON THE K-THEORY OF GROUPS WITH FINITE DECOMPOSITION COMPLEXITY

DANIEL KASPROWSKI

Abstract. It is proved that the assembly map in algebraic K- and L-theory with respect to the family of finite subgroups is injective for groups Γ with finite quotient finite decomposition complexity (a strengthening of finite decomposition complexity introduced by Guentner, Tesser and Yu) that admit a finite dimensional model for $\underline{E}\Gamma$ and have an upper bound on the order of their finite subgroups. In particular this applies to finitely generated linear groups over fields with characteristic zero with a finite dimensional model for $\underline{E}\Gamma$.

Introduction

Assembly maps in algebraic K- and L-theory are a useful tool to study the K- and L-theory of group rings $R[\Gamma]$. This is important for understanding geometric properties of manifolds with fundamental group Γ.

Finite decomposition complexity, introduced by Guentner, Tessera and Yu in [GTYa] and [GTYb], is a generalization of finite asymptotic dimension. In [RTY] the injectivity of the assembly map for algebraic K-theory was proved for groups Γ with finite decomposition complexity and finite classifying space $B\Gamma$. In this article we use the methods of [BR07b] to relax the finiteness assumption on $B\Gamma$ to allow for groups with torsion. To do so we need the stronger notion of finite quotient finite decomposition complexity. Our main result is the following theorem.

Theorem A. *Let Γ be a discrete group with finite quotient finite decomposition complexity and a global upper bound on the order of its finite subgroups and let R be a ring. Assume there exists a finite dimensional model for $\underline{E}\Gamma$. Then the assembly map*

$$H_*^\Gamma(\underline{E}\Gamma;\mathbb{K}_R) \to K_*(R[\Gamma]) \tag{0.1}$$

in algebraic K-theory is a split injection.

In Section 8 Theorem 8.1 is proved. This is a slightly stronger version than Theorem A. In Theorem 8.1 the assembly map is proved to be injective for every additive

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Γ-category $\mathcal{A}$. Theorem A follows from this by taking $\mathcal{A}$ to be the category of finitely generated free R-modules.
In section 9 a L-theoretic version of Theorem 8.1 is proved.

If Γ is torsion-free, then having finite quotient finite decomposition complexity is the same as having finite decomposition complexity. So we immediately get the following.

Corollary 1. *Let Γ be a discrete group with finite decomposition complexity that admits a finite dimensional model for $E\Gamma$ (in particular Γ is torsion-free). Then the assembly map* (0.1) *is split injective for every ring R.*

Since groups with finite asymptotic dimension have finite quotient finite decomposition complexity, we also get the following strengthening of Theorem A of [BR07b].

Corollary 2. *Let Γ be a discrete group with finite asymptotic dimension and a global upper bound on the order of its finite subgroups. Assume there exists a finite dimensional model for $\underline{E}\Gamma$, then the assembly map* (0.1) *is split injective for every ring R.*

In [BR07b] no assumption on the order of the finite subgroups is made but there is a gap in the proof of Proposition 7.4. Therefore, the proofs of the main results from [BR07b] are only correct under the additional assumption that there exists a cocompact model for $\underline{E}\Gamma$. In section 7 we prove a bounded version of the Descent Principle. This can be used to fix the proofs of the main theorems of [BR07b] under the additional assumption of an upper bound on the order of finite subgroups. If the proof of the Descent Principle 7.5 in [BR07b] can be fixed without further assumptions also our main theorem holds without the assumption on the finite subgroups.

In [GTYa] is was shown that linear groups have finite decomposition complexity and a slight modification of this proof will yield that they even have finite quotient finite decomposition complexity, as we will show in section 4. In [AS81] it was proved that a finitely generated subgroup of a linear group over a field of characteristic zero has a finite dimensional model for $\underline{E}\Gamma$ if and only if there is a global upper bound on the Hirsch rank of its unipotent subgroups. By Selbergs Lemma [Sel60] a finitely generated linear group over a field of characteristic zero is virtually torsion-free and thus has an upper bound on its finite subgroups. Therefore, we get the following corollary:

Corollary 3. *Let F be a field of characteristic zero, Γ a finitely generated subgroup of $GL_n(F)$ with a global upper bound on the Hirsch rank of its unipotent subgroups. Then the K-theoretic assembly map* (0.1) *is split injective for every ring R.*

A finitely generated linear group Γ over a field of positive characteristic has finite asymptotic dimension by [GTYb, Theorem 3.0.1] and $\underline{E}\Gamma$ admits a finite dimensional model by [DP, Corollary 5].

Corollary 4. *Let F be a field of positive characteristic, Γ a finitely generated subgroup of $GL_n(F)$. Suppose Γ has an upper bound on the order of its finite subgroups, then the K-theoretic assembly map* (0.1) *is split injective for every ring R.*

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Contents

Introduction 1
1. Metric properties of $\underline{E}\Gamma$ 4
2. Finite quotient Finite Decomposition Complexity 7
3. Permanence for fqFDC 8
4. Groups with fqFDC 10
5. Controlled Algebra 15
6. The Rips complex 18
7. The Descent Principle 20
8. Proof of the main theorem 25
9. L-theory 28
References 29

Notation.

- Γ will always denote a group. And all groups will be countable and discrete.
- Metrics are allowed to take on the value ∞ and a *metric* Γ*-space* will always be a metric space with an isometric (left) Γ-action.
- For a metric space X, a subspace $Y \subseteq X$ and $R > 0$ define
$$Y^R := \{x \in X \mid d(x, Y) < R\}.$$
- For metric spaces $\{(X_i, d_i)\}_{i \in I}$ we define
$$\coprod_{i \in I} X_i$$
to be the metric space with
$$d(x, y) = \begin{cases} d_i(x, y) & \text{if } \exists i : x, y \in X_i \\ \infty & \text{else} \end{cases}$$

1. Metric properties of $\underline{E}\Gamma$

In this section we will first show that any discrete, countable group Γ that admits a finite dimensional model for $\underline{E}\Gamma$ also admits a finite dimensional simplicial model with a proper Γ-invariant metric.

Definiton 1.1. A metric space (X, d) (resp. the metric d) is called *proper* if for every $R > 0$ and every $x \in X$ the closed ball $\bar{B}_R(x) := \{y \in X \mid d(x, y) \leq R\} \subseteq X$ is compact.
A metric d on X is called *finite* if $d(x, y) < \infty$ for all $x, y \in X$.

Lemma 1.2. *Let X be a finite dimensional Γ-CW-complex with countably many cells, then X is Γ-homotopy equivalent to a (countable) simplicial Γ-CW-complex of the same dimension.*

This lemma is stated in [Mis01] and is proved in [Hat02, 2C.5] for $\Gamma = \{e\}$. The construction from this proof can be done in an equivariant fashion.

Lemma 1.3. *Let X be a finite dimensional, countable (simplicial) Γ-CW-complex with finite stabilizers. Then X is Γ-homotopy equivalent to a locally finite, finite dimensional, countable (simplicial) Γ-CW-complex.*

Proof. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be an enumeration of the Γ-cells of X. For $t \geq 0$ let

$$Y_t := \{\sigma_n \mid n \leq \lfloor t \rfloor\}$$

and let X_t be the smallest subcomplex of X containing Y_t. Since Y_t contains only finitely many Γ-cells, also X_t is finite for every $t \geq 0$. The mapping telescope

$$T := \{(x, t) \in X \times [0, \infty) \mid x \in X_t\}$$

is a Γ-CW-complex that is homotopy equivalent to X and since X has only finite stabilizers T is locally finite.
If X is simplicial, then there is a simplicial structure on T with vertices (v, n), where $n \in \mathbb{N}$ and v a vertex of X_n. □

Lemma 1.4. *If X is a model for $\underline{E}\Gamma$, then X contains a countable Γ-subcomplex which is still a model for $\underline{E}\Gamma$.*

Proof. Because Γ has only countably many finite subgroups there exists a countable Γ-subcomplex $X_0 \subseteq X$ with fixed point sets $X^G \neq \emptyset$ for all $G \leq \Gamma$ finite.
Inductively define countable Γ-subcomplexes $X_i \subseteq X$ such that $X_{i-1}^G \hookrightarrow X_i^G$ is null homotopic for every finite subgroup $G \leq \Gamma$. Those exist because X^G is contractible and they can be chosen as countable complexes since the image of every contraction of X_{i-1}^G in X^G lies in a countable subcomplex.
Since $(\bigcup_{i\in\mathbb{N}} X_i)^G = \bigcup_{i\in\mathbb{N}} X_i^G$ and X_i^G is contractible in X_{i+1}^G, the subcomplex

$(\bigcup_{i\in\mathbb{N}} X_i)^G$ has vanishing homotopy groups and is therefore contractible. So $\bigcup_{i\in\mathbb{N}} X_i$ is a countable subcomplex of X which is still a model for $\underline{E}\Gamma$. ☐

By combining the three lemmas above we get:

Proposition 1.5. *If Γ admits a finite dimensional model for $\underline{E}\Gamma$, then it also admits a finite dimensional simplicial Γ-CW-model for $\underline{E}\Gamma$ with a proper Γ-invariant metric.*

Proof. By Lemma 1.4 there exists a countable, finite dimensional Γ-CW-model for $\underline{E}\Gamma$. So by Lemma 1.2 and 1.3 the group Γ admits a locally finite, finite dimensional, simplicial Γ-CW-model X for $\underline{E}\Gamma$. We can take the unique path metric on X which restricts to the Euclidean metric on every simplex. For a definition of the simplicial path metric see [RTY, Section 2]. This metric has the desired properties. ☐

Uniform contractibility. In the following we will need some contractibility properties. We will first define uniform contractibility with respect to subspaces and then prove the results about uniform contractibility from [BR07b] in the relative case.

Remark 1.6. Let G be a finite group, X a metric G-space and $q\colon X \to G\backslash X$ the quotient map. Then

$$d_{G\backslash X}(y, y') := d_X(q^{-1}(y), q^{-1}(y')) \quad \text{for } y, y' \in G\backslash X$$

defines a metric on $G\backslash X$. We will always consider quotients of metric spaces by an isometric action of a finite group as metric spaces using this metric.

In [BR07b, Definition 1.1] uniform $\mathcal{F}in$-contractibility was defined. We will need a relative version of this definition.

Definiton 1.7. Let X be a proper metric Γ-space, $Y \subseteq X$. X is said to be *uniformly $\mathcal{F}in$-contractible with respect to Y* if for every $R > 0$ there exists an $S > 0$ such that the following holds: For every $G \leq \Gamma$ finite and every G-invariant subset $B \subseteq X$ of diameter less than R with $B \cap Y^R \neq \emptyset$ the inclusion $B \cap Y^R \hookrightarrow B^S$ is G-equivariantly null homotopic.
We say that a metric space X is *uniformly contractible with respect to* $Y \subseteq X$ if for every $R > 0$ there is an $S > 0$ such that $B_R(x) \cap Y^R \hookrightarrow B_S(x)$ is null homotopic for every $x \in X$ with $B_R(x) \cap Y^R \neq \emptyset$.
Note that every metric space is uniformly contractible with respect to the empty subset.

In contrast to the definition in [BR07b] we require S to be independent of the finite subgroup G.
In [BR07b, Lemma 1.5] it was proved that if a group Γ admits a cocompact model X for $\underline{E}\Gamma$, then X is uniformly $\mathcal{F}in$-contractible. This can be modified to show the following relative version.

Lemma 1.8. *Let X be a Γ-CW-model for $\underline{E}\Gamma$ with a proper Γ-invariant metric. Then X is uniformly $\mathcal{F}in$-contractible with respect to every cocompact subset ΓK of X.*

Proof. Let $R > 0$ and ΓK cocompact be given. Because the metric on X is proper ΓK^R is contained in a cocompact subcomplex $\Gamma K'$ of X. Let k be an upper bound on the diameter of cells in $\Gamma K'$. Every subspace $Y \subseteq \Gamma K'$ (invariant under some $G \leq \Gamma$) of diameter less than R is contained in a finite subcomplex of $\Gamma K'$ (invariant under G) of diameter less than $R' := R + k$. Let $\mathcal{B}$ be the set of all finite subcomplexes of $\Gamma K'$ of diameter less than R'. Define $\mathcal{T} := \{(B, G) \mid B \in \mathcal{B}, G \leq \Gamma_B\}$, where $\Gamma_B := \{\gamma \in \Gamma \mid \gamma B = B\}$. Γ acts on $\mathcal{T}$ by $\gamma(B, G) := (\gamma B, G^\gamma)$, where $G^\gamma := \gamma G \gamma^{-1}$. Since $\Gamma K'$ is cocompact and the metric on $\Gamma K'$ is proper, the quotient $\mathcal{T}/\Gamma$ is finite. For every (B, G) there exists an $S = S(B, G) > 0$ such that B is G-equivariantly contractible in B^S since X is G-equivariantly contractible by assumption. Because the action of Γ on $\mathcal{T}$ has finite quotient, this S can be chosen independently of B and G. $\square$

Notation 1.9. Let X be a space with an action of a group Γ and let $\mathcal{S}$ be a collection of subsets of Γ. Define $X^{\mathcal{S}}$ to be the union of all fixed sets X^H, where $H \in \mathcal{S}$. If $\mathcal{S}$ is closed under conjugation by elements of Γ, then $X^{\mathcal{S}}$ is a Γ-invariant subspace of X.

Also [BR07b, Lemma 1.4] can be modified to a relative version.

Lemma 1.10. *Let X be a Γ-space with a proper Γ-invariant metric and $Y \subseteq X$ be a Γ-invariant subspace. For every finite subgroup $G \leq \Gamma$ let $J(G)$ be the set of families $\mathcal{S}$ of subgroups of G such that $\mathcal{S}$ is closed under conjugation by G. For all $n \in \mathbb{N}$ define*

$$J_n := \{(G, \mathcal{S}) \mid G \leq \Gamma, |G| \leq n, \mathcal{S} \in J(G)\}$$

and assume X is uniformly $\mathcal{F}in$-contractible with respect to Y. Then $\coprod_{(G,\mathcal{S}) \in J_n} G\backslash X^{\mathcal{S}}$ is uniformly contractible with respect to $\coprod_{(G,\mathcal{S}) \in J_n} G\backslash (Y \cap X^{\mathcal{S}})$ for every $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ and $R > 0$ be given. Let $S > 0$ be such that for every finite subgroup $G \leq \Gamma$ and every G-invariant subset $B \subseteq X$ of diameter less or equal to $2Rn$ with $B \cap Y^R \neq \emptyset$ the inclusion $B \cap Y^R \hookrightarrow B^S$ is G-equivariantly null homotopic. Let $(G, \mathcal{S}) \in J_n$, $y \in G\backslash X^{\mathcal{S}}$ and $x \in q^{-1}(y)$, where $q\colon X^{\mathcal{S}} \to G\backslash X^{\mathcal{S}}$ is the quotient map. Let H be the subgroup of G consisting of all $g \in G$ for which there are $g_1, .., g_m \in G$ such that $g_1 = e$, $g_m = g$ and $d(g_i x, g_{i+1} x) \leq 2R$. Then the diameter of $B := Hx^R \subseteq X$ is bounded by $2R|G| \leq 2Rn$. Therefore, if $B \cap Y^R \neq \emptyset$, the inclusion $B \cap Y^R \hookrightarrow B^S$ is H-equivariantly null homotopic. In particular let $z \in B^S$ be the point fixed by H on which $B \cap Y^R$ contracts (for some choice of a null homotopy). For $g \in G - H$ we have $gB \cap B = \emptyset$ and therefore the inclusion $GB \cap Y^R \hookrightarrow GB^S$ is G-equivariantly homotopic to a map that sends gB to gz. By G-equivariance this

homotopy can be restricted to $X^{\mathcal{S}}$, which induces the required null homotopy on the quotient. $\square$

2. Finite quotient Finite Decomposition Complexity

Definiton 2.1. Let $I := \{0,1\} \times \mathbb{N}^{>0}$. A metric space X decomposes over a class of metric spaces $\mathcal{C}$ if for every $r > 0$ there exists a decomposition $X = \bigcup_{(\epsilon,n)\in I} X_{\epsilon,n}$ with $d(X_{\epsilon,n_1}, X_{\epsilon,n_2}) > r$ if $n_1 \neq n_2$ and such that the metric space $\coprod_{(\epsilon,n)\in I} X_{\epsilon,n}$ is in $\mathcal{C}$.
A class of metric spaces $\mathcal{C}$ is stable under decomposition if every metric space which decomposes over $\mathcal{C}$ actually belongs to $\mathcal{C}$.

A metric space X is called *semi-bounded* if there exists $R > 0$ such that for all $x, y \in X$ either $d(x,y) < R$ or $d(x,y) = \infty$.

Definiton 2.2. The class of metric spaces $\mathcal{D}$ with *finite decomposition complexity* is the minimal class of metric spaces which contains all semi-bounded spaces and is stable under decomposition. We say X has FDC if it is contained in $\mathcal{D}$.

This definition of FDC is the same as the original definition ([GTYa, 2.1.3]) but using only metric spaces instead of metric families. A metric family $\{X_i\}$ has FDC in the sense of [GTYa] if and only if $\coprod X_i$ has FDC as in the above definition.

Definiton 2.3. Let X be a metric Γ-space and $Y \subseteq X$ a subspace. We say that Y has *finite quotient FDC* (fqFDC) if for every $n \in \mathbb{N}$ the space

$$\coprod_{G\leq\Gamma, |G|\leq n} G\backslash GY$$

has FDC.

Remark 2.4. As remarked in the introduction if [BR07b, Theorem 7.5] is true, then we do not need the assumption on the upper bound on the order of the finite subgroups of Γ in our main theorem. In that case it would also suffice that $G\backslash\Gamma$ has FDC for every finite subgroup of Γ (and a proper invariant metric on Γ) instead of assuming that $\coprod_{G\leq\Gamma,|G|\leq n} G\backslash\Gamma$ has FDC for every $n \in \mathbb{N}$.

Recall the following definition of asymptotic dimension. By [Roe03, Theorem 9.9] it is equivalent to the original definition of Gromov [Gro93].

Definiton 2.5. Let X be a metric space. The *dimension of a cover* $\mathcal{U}$ of X is the smallest integer n such that each $x \in X$ is contained in at most $n+1$ members of $\mathcal{U}$. The cover $\mathcal{U}$ is called *bounded* if $\sup_{U\in\mathcal{U}} \operatorname{diam} U < \infty$. $\mathcal{U}$ has *Lebesgue number* at least $R > 0$ if the ball of radius R around x is contained in some $U \in \mathcal{U}$ for each $x \in X$.

The *asymptotic dimension* of X is the smallest integer n such that for any $R > 0$ there exists an n-dimensional bounded cover $\mathcal{U}$ of X whose Lebesgue number is at least R.

Lemma 2.6. *Let X be a metric Γ-space with finite asymptotic dimension. Then X has fqFDC.*

Proof. Let $k := \operatorname{asdim} X$ and let $R > 0$ be given. Let $\mathcal{U}$ be an at most k-dimensional bounded cover of X whose Lebesgue number is at least R. For every $G \leq \Gamma$ finite let $p_G\colon X \to G\backslash X$ be the quotient map. Define $\mathcal{U}^G := \{p_G(U) \mid U \in \mathcal{U}\}$. Then $\{\mathcal{U}^G\}_{|G|\leq n}$ is a bounded cover of $\coprod_{G\leq\Gamma,|G|\leq n} G\backslash X$ with Lebesque number at least R and dimension at most $n(k+1)$. Therefore, $\coprod_{G\leq\Gamma,|G|\leq n} G\backslash X$ has asymptotic dimension at most $n(k+1)$.
By [GTYa, 4.1] a space with finite asymptotic dimension has FDC. □

3. Permanence for fqFDC

We begin by recalling some elementary concepts from coarse geometry.

Definiton 3.1. A map $f\colon X \to Y$ is

- *uniformly expansive* if there exists a non-decreasing function
$$\rho\colon [0,\infty) \to [0,\infty)$$
such that for every $x, y \in X$ with $d(x,y) < \infty$
$$d(f(x), f(y)) \leq \rho(d(x,y)),$$
- *effectively proper* if there exists a proper non-decreasing function
$$\delta\colon [0,\infty) \to [0,\infty)$$
such that for every $x, y \in X$ with $d(x,y) < \infty$
$$\delta(d(x,y)) \leq d(f(x), f(y)),$$
- a *coarse embedding* if it is both uniformly expansive and effectively proper and
- *metrically coarse* if it is uniformly expansive and proper. If X is proper and the metric on X is finite, then f is metrically coarse if it is a coarse embedding.

A metrically coarse homotopy between proper continuous maps is called a *metric homotopy*.

Remark 3.2.

- In the case where the metric spaces have a finite metric these definitions coincide with the common definitions but we allow effectively proper maps to map points with infinite distance close together.

- Let $X = \bigcup_{i\in I} X_i, Y = \bigcup_{j\in J} Y_j$, where different components have infinite distance and the metric restricted to each X_i resp. Y_i is finite. Then viewing X and Y as families of metric spaces $\{X_i\}_{i\in I}, \{Y_j\}_{j\in J}$ the above definitions coincide with those from [GTYb].

Recall the following permanence properties of FDC from [GTYa]. We will restate them using only spaces instead of families.

Lemma 3.3 (Coarse Invariance [GTYa, 3.1.3]). *Let X, Y be metric spaces. If there is a coarse embedding from X to Y and Y has FDC, then so does X. In particular if X has FDC, then for any family of subspaces $\{X_i\}_{i\in I}$ the space $\coprod_{i\in I} X_i$ has FDC.*

Theorem 3.4 (Fibering [GTYa, 3.1.4]). *Let X, Y be metric spaces and $f\colon X \to Y$ uniformly expansive. Assume Y has finite decomposition complexity and that for every $R > 0$ and every family of subspaces $\{Z_i\}_{i\in I}$ of Y with $\operatorname{diam} Z_i \leq R$ for all $i \in I$ also the space $\coprod_{i\in I} f^{-1}(Z_i)$ has FDC. Then X has FDC.*

Theorem 3.5 (Finite Union [GTYa, 3.1.7]). *Let X be a metric space expressed as a union $X = \bigcup_{i=1}^n X_i$ of finitely many subspaces. If the space $\coprod_{i=1}^n X_i$ has FDC, so does X.*

The following are generalizations to fqFDC of the above permanence properties.

Lemma 3.6 (Coarse Invariance). *Let X, Y be metric Γ-spaces. If there is a Γ-equivariant coarse embedding from X to Y and Y has fqFDC, then so does X. In particular if X has fqFDC, then for any family of subspaces $\{X_i\}_{i\in I}$ the space $\coprod_{i\in I} X_i$ has fqFDC.*

Proof. Let ρ, δ be given as in Definition 3.1. Then for all finite $G \leq \Gamma$ the map $\bar{f}\colon G\backslash X \to G\backslash Y$ fulfills the following for all $\bar{x}, \bar{y} \in G\backslash X$:

$$d(\bar{f}(\bar{x}), \bar{f}(\bar{y})) = \min_{g\in G} d(f(x), gf(y)) \leq \min_{g\in G} \rho(d(x, gy)) = \rho(d(\bar{x}, \bar{y}))$$

and

$$\delta(d(\bar{x}, \bar{y})) = \min_{g\in G} \delta(d(x, gy)) \leq \min_{g\in G} d(f(x), f(gy)) = d(\bar{f}(\bar{x}), \bar{f}(\bar{y})).$$

So for every $k > 0$ the map

$$F\colon \coprod_{G\leq\Gamma, |G|\leq k} G\backslash X \to \coprod_{G\leq\Gamma, |G|\leq k} G\backslash Y$$

is a coarse embedding as well. Since $\coprod_{G\leq\Gamma, |G|\leq k} G\backslash Y$ has FDC by assumption, so does $\coprod_{G\leq\Gamma, |G|\leq k} G\backslash X$ by Lemma 3.3. □

Lemma 3.7 (Fibering)**.** *Let X, Y be metric Γ-spaces, $f\colon X \to Y$ uniformly expansive and Γ-equivariant. Assume Y has fqFDC and for all $R > 0$ the space $\coprod_{y\in Y} f^{-1}(B_R(y))$ has fqFDC (as a subspace of $\coprod_{y\in Y} X$ with componentwise Γ-action). Then X has fqFDC.*

Proof. As above the induced map $F\colon \coprod_{|G|\le k} G\backslash X \to \coprod_{|G|\le k} G\backslash Y$ is uniformly expansive and $\coprod_{|G|\le k} G\backslash Y$ has FDC by assumption. By Theorem 3.4 it suffices to show that for every $R > 0$ and every family $\{Z_i\}_{i\in I}$ of subspaces of $\coprod_{|G|\le k} G\backslash Y$ with $\operatorname{diam} Z_i < R$ for all $i \in I$ the space $\coprod_{i\in I} F^{-1}(Z_i)$ has FDC. Because of the bound on the diameter for all $i \in I$ there exists a $G_i \le \Gamma$ with $|G_i| \le k$ and $Z_i \subseteq G_i\backslash Y$. For all $i \in I$ let $pr_i\colon Y \to G_i\backslash Y$ be the projection and choose $y_i \in pr_i^{-1}(Z_i)$. Define $Z_i' := pr_i^{-1}(Z_i) \cap B_R(y_i)$. Then $pr_i(Z_i') = Z_i$ and $\operatorname{diam} Z_i' < R$.
Let $pr_i'\colon X \to G_i\backslash X$ be the projection and $f_i\colon G_i\backslash X \to G_i\backslash Y$ the map induced by f. Then

$$pr_i'(f^{-1}(Z_i')) = f_i^{-1}(pr_i(Z_i')) = f_i^{-1}(Z_i),$$

and since $\coprod_{i\in I} f^{-1}(Z_i') \subseteq \coprod_{y\in Y} f^{-1}(B_R(y))$ has fqFDC by assumption, in particular $\coprod_{i\in I} pr_i'(f^{-1}(Z_i')) = \coprod f_i^{-1}(Z_i)$ has FDC. But this is the same as $\coprod_{i\in I} F^{-1}(Z_i)$. □

Lemma 3.8 (Finite Union)**.** *Let X be a metric Γ-space, $Y \subseteq X$, for $i \in \{0,..,n\}$ let $X_i \subseteq X$ have fqFDC and assume $Y \subseteq \bigcup_{i=1}^n X_i$. Then Y has fqFDC.*

Proof. In the case $n = 2$ the space $\coprod_{|G|\le n} G\backslash GY$ decomposes over a subspace of $\left(\coprod_{|G|\le n} G\backslash GX_1\right) \amalg \left(\coprod_{|G|\le n} G\backslash GX_2\right)$, which has FDC by assumption. Therefore, $\coprod_{|G|\le n} G\backslash GY$ has FDC. The general case follows by induction. □

Remark 3.9. It is well known that also a finite union of spaces with finite asymptotic dimension has again finite asymptotic dimension, see [Roe03, Proposition 9.13].

4. Groups with fqFDC

On every group Γ there exists a finite proper left invariant metric. For any two finite proper left invariant metrics d, d' on Γ the identity map $(\Gamma, d) \to (\Gamma, d')$ is a coarse embedding. Therefore, the following definition does not depend on the chosen metric.

Definiton 4.1. A group Γ has *fqFDC* if (Γ, d) has fqFDC for a finite proper left invariant metric d.

Remark 4.2. For any subgroup $G \le \Gamma$ and any finite proper left invariant metric d on Γ the function

$$d_{G\backslash\Gamma}(G\gamma, G\gamma') := \inf_{g\in G} d(g\gamma, \gamma') = \min_{g\in G} d(g\gamma, \gamma')$$

defines a finite proper metric on $G\backslash\Gamma$.
If we have a finite proper left invariant metric on Γ and a normal subgroup $K \trianglelefteq \Gamma$, we will always consider this metric on Γ/K. This metric is again left invariant.
If we talk about fqFDC in context of a group Γ, we will always mean Γ with a chosen finite proper left invariant metric.

To prove some permanence properties for groups we first need a stronger version of fqFDC.

Definiton 4.3. A group Γ has *strong fqFDC* if for all groups Γ' which contain Γ as a normal subgroup and all $k \in \mathbb{N}$ the space $\coprod_{H\leq\Gamma',|H|\leq k} H\backslash H\Gamma$ has FDC.

Lemma 4.4. *Strong fqFDC is closed under extensions $K \to \Gamma \to Q$ where K is a characteristic subgroup of Γ.*

Proof. Let $K \trianglelefteq \Gamma$ be characteristic, $Q := \Gamma/K$ and $\Gamma \trianglelefteq \Gamma'$. Assume K and Q have strong fqFDC.
Since K is a characteristic subgroup of Γ the group K is normal in Γ' and Q is normal in Γ'/K. So we get a uniformly expansive map

$$\coprod_{H\leq\Gamma',|H|\leq k} H\backslash H\Gamma \longrightarrow \coprod_{H\leq\Gamma',|H|\leq k} (H/H\cap K)\backslash(H/H\cap K)Q$$

and $\coprod_{H\leq\Gamma',|H|\leq k}(H/H\cap K)\backslash(H/H\cap K)Q$ has FDC because Q has strong fqFDC. So by Theorem 3.4 it suffices to show that for all $r > 0$ the space $\coprod_{|H|\leq k,q\in\Gamma} H\backslash HqB_r(e)K$ has FDC, where $B_r(e)K = \{\gamma k \mid \gamma \in B_r(e), k \in K\}$. We have

$$\coprod_{|H|\leq k,q\in\Gamma} H\backslash HqB_r(e)K \subseteq \bigcup_{\gamma\in B_r(e)} \coprod_{|H|\leq k,q\in\Gamma} H\backslash Hq\gamma K = \bigcup_{\gamma\in B_r(e)} \coprod_{|H|\leq k,q\in\Gamma} q\gamma H^{q\gamma}\backslash H^{q\gamma}K$$

where $H^{q\gamma} = (q\gamma)^{-1}Hq\gamma$. Now the lemma follows by Theorem 3.5 and the assumption that K has strong fqFDC. $\square$

Lemma 4.5. *Let Γ be the direct union of groups Γ_i having finite asymptotic dimension. Than Γ has strong fqFDC.*

Proof. Let $k \in \mathbb{N}$ and $\Gamma \trianglelefteq \Gamma'$ be given. For any $r > 0$ and $H \leq \Gamma'$ with $|H| \leq k$ define

$$Z_H := \langle B_r(e) \cap H\Gamma\rangle.$$

Since

$$H\Gamma = \coprod_{\gamma Z_H\in H\Gamma/Z_H}^{r-disj} \gamma Z_H,$$

in particular

$$H\Gamma = \coprod_{H\gamma Z_H \in H\backslash H\Gamma/Z_H}^{r-disj} H\gamma Z_H.$$

Therefore, $\coprod_{|H|\leq k} H\backslash H\Gamma$ r-decomposes over $\coprod_{\gamma\in\Gamma,|H|\leq k} H\backslash H\gamma Z_H$.
So it suffices to show that $\coprod_{\gamma\in\Gamma,|H|\leq k} H\backslash H\gamma Z_H$ has FDC for any $r > 0$.
Let i be such that $B_{2(k+1)r}(e) \cap \Gamma \subseteq \Gamma_i$.

Claim: $Z_H \cap \Gamma \leq \Gamma_i$ for all $|H| \leq k$.

Using this we conclude that $\coprod_{|H|\leq k} Z_H \cap \Gamma$ has finite asymptotic dimension. Furthermore, $Z_H/Z_H \cap \Gamma \leq H\Gamma/\Gamma \cong H/H \cap \Gamma$ has less or equal to k elements. For every H choose $h_i^H \in Z_H$ with $Z_H = \bigcup_{i=1}^k h_i^H(Z_H \cap \Gamma)$. Since $h_i^H(Z_H \cap \Gamma)$ is isometric to $Z_H \cap \Gamma$ for every $1 \leq i \leq k$ the space $\coprod_{|H|\leq k} h_i^H(Z_H \cap \Gamma)$ has finite asymptotic dimension and, therefore, also

$$\coprod_{|H|\leq k} Z_H = \bigcup_{i=1}^{k} \coprod_{|H|\leq k} h_i^H(Z_H \cap \Gamma)$$

has finite asymptotic dimension.
Now numerating the elements of each H with $|H| \leq k$ we conclude in the same way that

$$\coprod_{\gamma\in\Gamma,|H|\leq k} H\gamma Z_H = \bigcup_{i=1}^{k} \coprod_{\gamma\in\Gamma,|H|\leq k} h_i^H\gamma Z_H$$

has finite asymptotic dimension.
If $\operatorname{asdim} \coprod_{\gamma\in\Gamma,|H|\leq k} H\gamma Z_H \leq m$, then $\operatorname{asdim} \coprod_{\gamma\in\Gamma,|H|\leq k} H\backslash H\gamma Z_H \leq k(m+1)$ (see the proof of Lemma 2.6).
Therefore, $\coprod_{\gamma\in\Gamma,|H|\leq k} H\backslash H\gamma Z_H$ has FDC.

It remains to prove the above claim:
Let $z \in Z_H$ and let m be minimal with

$$z = h_1\gamma_1 \dots h_m\gamma_m\gamma$$

for some $h_j \in H, \gamma_j \in \Gamma, \gamma \in \Gamma_i$ such that $h_j\gamma_j \in B_r(e)$ for all j.
Assume $m > k \geq |H|$, then there exist $m - |H| \leq n_1 < n_2 \leq m$ with

$$h_{n_1} \dots h_m = h_{n_2} \dots h_m.$$

Therefore,

$$(h_{n_1}\gamma_{n_1} \dots h_m\gamma_m)^{-1} h_{n_2}\gamma_{n_2} \dots h_m\gamma_m \in \Gamma \cap B_{2(k+1)r}(e) \subseteq \Gamma_i$$

and there exists $\gamma' \in \Gamma_i$ with

$$h_{n_1}\gamma_{n_1} \cdot \ldots \cdot h_m\gamma_m = h_{n_2}\gamma_{n_2} \cdot \ldots \cdot h_m\gamma_m\gamma'.$$

So m is not minimal, a contradiction.
Let $z \in Z_H \cap \Gamma$ be represented as

$$z = h_1\gamma_1 \ldots h_m\gamma_m\gamma$$

for some $h_j\gamma_j \in B_r(e) \cap H\Gamma, \gamma \in \Gamma_i$ with $m \leq k$.
Then $h_1\gamma_1 \ldots h_m\gamma_m \in \Gamma \cap B_{kr}(e) \subseteq \Gamma_i$ and therefore also $z = h_1\gamma_1 \ldots h_m\gamma_m\gamma \in \Gamma_i$. This proves the claim. □

By the classification of finitely generated abelian groups we immediately get the following:

Corollary 4.6. *Abelian groups have strong fqFDC.* □

Combining Lemma 4.4 and Corollary 4.6 yields:

Corollary 4.7. *Solvable groups have strong fqFDC.* □

To show that finitely generated linear groups have fqFDC we need the following extension property.

Proposition 4.8. *Let $K \to \Gamma \to Q$ be an extension and let K have strong fqFDC and Q fqFDC. Then Γ has fqFDC.*

Proof. $\Gamma \to Q$ is uniformly expansive and Q has fqFDC. So by Lemma 3.7 it suffices to show that for all $r > 0$ the space

$$\coprod_{g\in\Gamma} gB_r(e)K \subseteq \bigcup_{\gamma\in B_r(e)} \coprod_{g\in\Gamma} g\gamma K$$

has fqFDC. This follows from Lemma 3.8 and the fact that for every $k \in \mathbb{N}, \gamma \in B_r(e)$ the space

$$\coprod_{|H|\leq k, g\in\Gamma} H\backslash Hg\gamma K = \coprod_{|H|\leq k, g\in\Gamma} g\gamma H^{g\gamma}\backslash H^{g\gamma} K$$

has FDC by assumption. □

Linear groups.

Theorem 4.9. *Let Γ be a finitely generated subgroup of $GL_n(F)$ where F is a field. Then Γ has fqFDC.*

This theorem is the fqFDC version of [GTYb, 3.0.1]. All steps in the proof hold for fqFDC as well, we only need to proof the following fqFDC version of [GTYb, 3.3.1].

Lemma 4.10. *Let G be a countable discrete group. Suppose there exists a (pseudo-) length function l' on G with the following properties:*

(a) *G has finite asymptotic dimension with respect to the associated (pseudo-) metric d'.*
(b) *For all $r > 0$ there exists l_r, a (pseudo-) length function on G, for which*
    (i) *G has finite asymptotic dimension with respect to the associated (pseudo-) metric d_r,*
    (ii) *l_r is proper when restricted to $B_{r,d'}(e)$, the ball of radius r around e with respect to the metric d'.*

Then G has fqFDC.

Condition (ii) in the lemma means precisely that $B_{s,d_r}(e) \cap B_{r,d'}(e)$ is finite for every $s > 0$.
In the proof we will use that finite asymptotic dimension implies fqFDC (see Lemma 2.6).

Proof. Fix a proper length function l on G with associated metric d. By Lemma 3.7, applied to the identity map $(G, d) \to (G, d')$, it suffices to show that for every $r > 0$ the space $\coprod_{g\in G} B_{r,d'}(g) = \coprod_{g\in G} gB_{r,d'}(e)$ has finite asymptotic dimension when equipped with the metric d. Because all spaces $gB_{r,d'}(e)$ are isometric to $B_{r,d'}(e)$ it suffices to show that $B_{r,d'}(e)$ has finite asymptotic dimension.
Let $r > 0$. Pick l_{2r} as in the assumptions. The ball $B_{r,d'}(e) \subseteq G$ has finite asymptotic dimension with respect to the metric d_{2r}.
Thus, it remains to show that the metrics d and d_{2r} on $B_{r,d'}(e)$ are coarsely equivalent. Since l-balls in G are finite, we easily see that for every s there exists s' such that if $d(g, h) \leq s$, then $d_{2r}(g, h) \leq s'$; this holds for every g and h in G. Conversely, for every s the set $B_{2r,d'}(e)\cap B_{s,d_{2r}}(e)$ is finite by assumption, and we obtain s' such that for every g in this set $l(g) \leq s'$. If now $g, h \in B_{r,l'}(e)$ are such that $d_{2r}(g, h) \leq s$, then $g^{-1}h \in B_{s,d_{2r}}(e)$ and $d(g, h) = l(g^{-1}h) \leq s'$. □

To generalize this to arbitrary commutative rings we need Lemma 5.2.3 from [GTYa]:

Lemma 4.11. *Let R be a finitely generated commutative ring with unit and let $\mathfrak{n}$ be the nilpotent radical of R,*

$$\mathfrak{n} = \{r \in R \mid \exists n : r^n = 0\}.$$

The quotient ring $S = R/\mathfrak{n}$ contains a finite number of prime ideals $\mathfrak{p}_1, .., \mathfrak{p}_n$ such that the diagonal map

$$S \to S/\mathfrak{p}_1 \oplus \ldots \oplus S/\mathfrak{p}_n$$

embeds S into a finite direct sum of domains.

The next theorem is the fqFDC version of [GTYa, 5.2.2]. We need to assume that Γ is finitely generated because we do not know if fqFDC is closed under unions.

Theorem 4.12. *Let R be a commutative ring with unit and let Γ be a finitely generated subgroup of $GL(n,R)$, then Γ has fqFDC.*

Proof. Because Γ is finitely generated we can assume that R is finitely generated as well. With $\mathfrak{n}$ and S as in the previous lemma, we have an exact sequence

$$1 \to I + M_n(\mathfrak{n}) \to GL_n(R) \to GL_n(S)$$

in which $I + M_n(\mathfrak{n})$ is nilpotent and therefore has strong fqFDC by Corollary 4.7. In the notation of the previous lemma we have embeddings

$$GL_n(S) \hookrightarrow GL_n(S/\mathfrak{p}_1) \times \ldots \times GL_n(S/\mathfrak{p}_n) \hookrightarrow GL_n(Q(S/\mathfrak{p}_1)) \times \ldots \times GL_n(Q(S/\mathfrak{p}_n))$$

where $Q(S/\mathfrak{p}_i)$ is the quotient field of $S/\mathfrak{p}_i$.
So the quotient has fqFDC by Theorem 4.9.
Now the theorem follows from Proposition 4.8. ∎

5. Controlled Algebra

Let X be a proper metric space, $\mathcal{A}$ a small additive category and Γ a group. The definition of a geometric module in this article is a slight variation of the definitions in [BR07b] and [RTY]. The first definition of geometric groups appeared in [CH69] and of geometric modules in [Qui79] and [Qui82]. The first definition of continuous control is in [ACFP94].

Definiton 5.1. Let $Z := \Gamma \times X \times [0,1)$. A *geometric $\mathcal{A}$-module* M over X is given by a sequence of objects $(M_z)_{z\in Z}$ in $\mathcal{A}$, subject to the following conditions:

(a) The image of $\operatorname{supp}(M) = \{z \in Z \mid M_z \neq 0\}$ under the projection
$$Z \to X \times [0,1)$$
is locally finite.

(b) For every $x \in X, t \in [0,1)$ the set $\operatorname{supp}(M) \cap (\Gamma \times \{x\} \times \{t\})$ is finite.

A *morphism* $\varphi\colon M \to N$ between geometric modules M, N is a sequence $(\varphi_{x,y}\colon M_y \to N_x)_{(x,y)\in Z^2}$ of morphisms in $\mathcal{A}$, subject to the following conditions:

(a) φ is *continuously controlled at 1*, i.e. for each $x \in X$ and each neighborhood U of $(x,1)$ in $X \times [0,1]$ there exists a neighborhood V of $(x,1)$ in $X \times [0,1]$ such that for all $\gamma,\gamma' \in \Gamma, v \in V, y \notin U$, $\varphi_{(\gamma,v),(\gamma',y)} = \varphi_{(\gamma',y),(\gamma,v)} = 0$.

(b) For every $z \in Z$ the set $\{z' \in Z \mid \varphi_{z,z'} \neq 0 \text{ or } \varphi_{z',z} \neq 0\}$ is finite.

(c) φ is *R-bounded* for some $R > 0$, i.e. $d(x,x') > R$ implies $\varphi_{(\gamma,x,t),(\gamma',x',t')} = 0$ for all $\gamma,\gamma' \in \Gamma, x,x' \in X, t,t' \in [0,1)$.

Let $\mathcal{A}_\Gamma(X)$ denote the category of geometric $\mathcal{A}$-modules over X and their morphisms. The composition of morphisms is given by matrix multiplication. $\mathcal{A}_\Gamma(X)$ is an additive category with pointwise addition.

Remark 5.2. Let $\mathcal{A}_c(X) \subseteq \mathcal{A}_\Gamma(X)$ be the full additive subcategory with objects having support in $\{e\} \times X \times [0,1)$. This coincides with the definition of $\mathcal{A}_c(X)$ in [RTY].
The inclusion $\mathcal{A}_c(X) \hookrightarrow \mathcal{A}_\Gamma(X)$ is an equivalence because of condition (b) on the objects of $\mathcal{A}_\Gamma(X)$.

We now recall the definition of a Karoubi filtration [CP95, Definition 1.27].

Definiton 5.3. Let $\mathcal{U}$ be a full subcategory of an additive category $\mathcal{A}$. We say that $\mathcal{A}$ is $\mathcal{U}$-filtered if every object $A \in \mathcal{A}$ has a family of decompositions $\{A \cong E_i \oplus U_i\}$ with $E_i \in \mathcal{A}, U_i \in \mathcal{U}$ such that

(a) for each $A \in \mathcal{A}$ the decomposition forms a filtered poset under the partial order $E_i \oplus U_i \leq E_j \oplus U_j$ whenever E_j is a direct summand of E_i and U_i is a direct summand of U_j.
(b) for every $A \in \mathcal{A}, U \in \mathcal{U}$ every map $f\colon A \to U$ factors as
$$A \cong E_i \oplus A_i \to E_i \to A$$
for some i.
(c) for every $A \in \mathcal{A}, U \in \mathcal{U}$ every map $f\colon U \to A$ factors as
$$U \to U_i \to E_i \oplus U_i \cong A$$
for some i.
(d) for each $A, B \in \mathcal{A}$ the filtration of $A \oplus B$ is equivalent to the sum of the filtrations $\{A = E_i \oplus U_i\}$ and $\{B = F_j \oplus V_j\}$, i.e. to $\{E_i \oplus F_j \oplus U_i \oplus V_j\}$.

The quotient $\mathcal{A}/\mathcal{U}$ is defined as the category having the same objects as $\mathcal{A}$ and morphism equivalence classes of morphism of $\mathcal{A}$ where $f, g\colon A \to B$ are equivalent if $f - g$ factors through some $U \in \mathcal{U}$.

By K-theory of an additive category we will always mean the non-connective K-theory spectrum [PW85].

Theorem 5.4 ([CP95, Theorem 1.28])**.** *If $\mathcal{A}$ is $\mathcal{U}$-filtered, then*
$$\mathbb{K}(\mathcal{U}) \to \mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{A}/\mathcal{U})$$
is a homotopy fibration sequence.

Definiton 5.5 ([PW85, Definition 1.1])**.** An additive category $\mathcal{A}$ is said to be *filtered* if there is an increasing filtration
$$F_0(A,B) \subseteq F_1(A,B) \subseteq \ldots \subseteq F_n(A,B) \subseteq \ldots$$

on $\hom(A, B)$ for every pair of objects $A, B \in \mathcal{A}$. Each $F_i(A, B)$ has to be an additive subgroup of $\hom(A, B)$ and we must have $\bigcup_{i\in\mathbb{N}} F_i(A, B) = \hom(A, B)$. We require the zero and identity maps to be in the zeroth filtration degree and for $f \in F_i(A, B)$ and $g \in F_j(B, C)$ the composition $g \circ f$ to be in $F_{i+j}(A, C)$. If $f \in F_i(A, B)$, we say that f has *(filtration) degree* i.

Remark 5.6. We do not demand projections $A \oplus B \to A$ and inclusions $A \to A \oplus B$ to have degree zero because then we either have to specify choices for the sums or every isomorphism would have degree zero.

Definiton 5.7. For filtered additive categories $\{\mathcal{A}_i\}_{i\in I}$ we define $\prod^{bd}_{i\in I} \mathcal{A}_i$ to be the subcategory of $\prod_{i\in I} \mathcal{A}_i$ containing all objects and those morphisms $\varphi = \{\varphi_i\}_{i\in I}$ such that there exists $n \in \mathbb{N}$ with φ_i has degree n for all $i \in I$.
For a metric space X the categories $\mathcal{A}_\Gamma(X)$ and $\mathcal{A}_c(X)$ are filtered by defining a morphism f to be of degree n if it is n-bounded.

Proposition 5.8. *Let $\{Y_i \subseteq X_i\}_{i\in I}$ be a family of metric spaces with subspaces, then the inclusion $\prod^{bd}_{i\in I} \mathcal{A}_\Gamma(Y_i) \hookrightarrow \prod^{bd}_{i\in I} \mathcal{A}_\Gamma(X_i)$ is a Karoubi filtration. We will denote the quotient by $\prod^{bd}_{i\in I} \mathcal{A}_\Gamma(X_i, Y_i)$. The same holds when $\mathcal{A}_\Gamma$ is replaced by $\mathcal{A}_c$.*

Proof. Let $Z_i := \Gamma \times X_i \times [0, 1)$ and let $\varphi = \{\varphi_i : A_i \to U_i\}_{i\in I}$ be a morphism from $\{A_i\} \in \prod^{bd}_{i\in I} \mathcal{A}_\Gamma(X_i)$ to $\{U_i\} \in \prod^{bd}_{i\in I} \mathcal{A}_\Gamma(Y_i)$. Under the projection

$$p_i : Z_i \times Z_i \to X_i \times X_i$$

the subspace $\operatorname{supp}\varphi_i := \{(z, z') \in Z_i \times Z_i \mid \varphi_{z',z} \neq 0\}$ has image inside$X_i \times Y_i$. Let $V_i := \{x \in X_i \mid \exists y : (x, y) \in p_i(\operatorname{supp}\varphi_i)\}$. Define $\rho_i : V_i \to Y_i$ by choosing for every $v \in V_i$ a y with $(v, y) \in p_i(\operatorname{supp}\varphi_i)$. Furthermore, define

$$(U_i^\varphi)_y := \bigoplus_{x\in\rho_i^{-1}(y)} (A_i)_x.$$

This defines for every morphism starting from $\{A_i\}$ a decomposition

$$\{A_i \cong U_i^\varphi \oplus A_i|_{X_i\setminus V_i}\}_{i\in I}$$

and $\{U_i^\varphi\}_{i\in I} \cong \{A_i|_{V_i}\}_{i\in I}$, $\{U_i^\varphi\} \in \prod^{bd}_{i\in I} \mathcal{A}_\Gamma(Y_i)$. Taking these decompositions for every A indexed by all morphisms starting in A one can easily check that this gives a Karoubi filtration of $\prod^{bd}_{i\in I} \mathcal{A}_\Gamma(X_i)$. $\square$

Definiton 5.9. If Γ acts on X and $\mathcal{A}$, then Γ acts on the category $\mathcal{A}_\Gamma(X)$ by $(\gamma M)_x := \gamma(M_{\gamma^{-1}x})$ and the corresponding action on the morphisms. For a subgroup $G \leq \Gamma$ and $Y_i \subseteq X_i$ G-invariant let $\prod^{bd}_{i\in I} \mathcal{A}^G_\Gamma(X_i, Y_i)$ be the corresponding fixed point category of $\prod^{bd}_{i\in I} \mathcal{A}_\Gamma(X_i, Y_i)$. This is equivalent to the quotient of $\prod^{bd}_{i\in I} \mathcal{A}^G_\Gamma(X_i)$ by $\prod^{bd}_{i\in I} \mathcal{A}^G_\Gamma(Y_i)$.

Definiton 5.10. For a continuous map $p\colon X \to X'$ let $\mathcal{A}_{\Gamma,p}(X)$ be the full subcategory of $\mathcal{A}_\Gamma(X)$ with objects having support in $\Gamma \times p^{-1}(K) \times [0,1)$ for some compact subspace $K \subseteq X'$. For some $G \leq \Gamma$ acting on X let $\mathcal{A}^G_{\Gamma,p}(X)$ be the fixed point category as above.
Furthermore, let $\mathcal{A}^G_{\Gamma,p}(X)_0$ be the full subcategory of $\mathcal{A}^G_{\Gamma,p}(X)$ with the following condition on the support of the objects:

- For every object M the limit points of the image of $\mathrm{supp}(M)$ under $Z \to X \times [0,1)$ are disjoint from $X \times \{1\}$.

The inclusion $\mathcal{A}^G_{\Gamma,p}(X)_0$ into $\mathcal{A}^G_{\Gamma,p}(X)$ is a Karoubi filtration.
Define $\mathcal{A}^G_{\Gamma,p}(X)^\infty$ to be the quotient of $\mathcal{A}^G_{\Gamma,p}(X)$ by $\mathcal{A}^G_{\Gamma,p}(X)_0$.

The categories defined above are functorial in X for continuous metrically coarse maps $f\colon X \to Y$ respectively commutative diagrams of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[d, "p'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

where f is continuous metrically coarse and f' is continuous. If two such maps are metrically homotopic (see Section 3), then they induce homotopic maps

$$\mathbb{K}(\mathcal{A}^G_{\Gamma,p}(X)) \to \mathbb{K}(\mathcal{A}^G_{\Gamma,p'}(Y))$$

of the K-theory spectra ([BR07b, (5.15)]).

Remark 5.11. Let $p\colon \underline{E}\Gamma \to \Gamma\backslash\underline{E}\Gamma$ be the projection. The category $\mathcal{A}^\Gamma_{p,\Gamma}(\underline{E}\Gamma)_0$ is equivalent to the category $\mathcal{A}[\Gamma]$ defined in [BR07a]. There also is the assembly map

$$H_*(\underline{E}\Gamma; \mathbb{K}_\mathcal{A}) \to H_*(pt; \mathbb{K}_\mathcal{A}) = \pi_*\mathbb{K}(\mathcal{A}[\Gamma])$$

defined.
The controlled categories defined above can be used to describe that assembly map. More precise the boundary map

$$\pi_*\mathbb{K}\mathcal{A}^\Gamma_{p,\Gamma}(\underline{E}\Gamma)^\infty \to \pi_{*-1}\mathbb{K}\mathcal{A}^\Gamma_{p,\Gamma}(\underline{E}\Gamma)_0$$

is equivalent to the assembly map. Compare [BR07b, (5.17)].

6. The Rips complex

Recall the following definitions.

Definiton 6.1. A metric space X has *bounded geometry* if for each $R > 0$ there exists $N > 0$ such that for all $x \in X$ the ball $B_R(x)$ contains at most N points.

Definiton 6.2. Given a metric space X and a number $s > 0$, the *Rips complex* $P_s(X)$ is the simplicial complex with vertex set X and with a simplex $\langle x_0, \dots, x_n\rangle$ whenever $d(x_i, x_j) \leq s$ for all $i, j \in \{0, ..., n\}$.

Note that if X is a metric space with bounded geometry, then the Rips complex $P_s(X)$ is finite dimensional and locally finite. We will always use the simplicial path metric on $P_s(X)$. For a definition of the simplicial path metric see [RTY, Section 2]. The following is similar to [GTYb, 4.3.6] and is proved in a special case in the proof of [RTY, Theorem 7.8]

Proposition 6.3. *Let X be a metric CW-complex, Y a subcomplex such that X is uniformly contractible with respect to Y and the cells in Y have uniformly bounded diameter. Let S be a subspace with bounded geometry such that there exists $R > 0$ with $Y \subseteq S^R$. Then there exist continuous metrically coarse maps $f_s \colon Y \to P_s(S)$, $g_s \colon P_s(S) \to X$ for every $s > 2R$ such that the following diagram commutes for $s' > s$*

$$\begin{tikzcd} Y \arrow[r, "f_s"] \arrow[dr, "f_{s'}"'] & P_s(S) \arrow[r, "g_s"] \arrow[d, "i_{ss'}"] & X \\ & P_{s'}(S) \arrow[ur, "g_{s'}"'] & \end{tikzcd}$$

where $i_{ss'} \colon P_s(S) \to P_{s'}(S)$ is the inclusion and such that $g_s \circ f_s$ is metrically homotopic to the inclusion $i \colon Y \hookrightarrow X$.

Proof. Let $R > 0$ be such that $Y \subseteq S^R$, then $\{U_{x'} := Y \cap (B_R(x') - (S - \{x'\}))\}_{x' \in S}$ is an open covering of Y. Choose a partition of unity $\{\varphi_{x'}\}$ subordinate to the cover $\{U_{x'}\}$ and define $f_s \colon Y \to P_s(S)$ by

$$f_s(y) := \sum_{x' \in S} \varphi_{x'}(y) x'$$

for every $y \in Y$, $s > 2R$.
Define maps $g_s \colon P_s(S) \to X$ by induction over $s \in \mathbb{N}$ and the simplices in $P_s(S)$. If $s = 0$, $P_s(S) = S$ and g_0 is just the embedding $S \hookrightarrow X$.
Now assume g_{s-1} has been defined. Let $P_s^{(k)}(S)$ be the k-skeleton of $P_s(S)$. Viewing $P_{s-1}(S)$ as a subcomplex of $P_s(S)$ we extend g_{s-1} inductively over the subspaces $P_s^{(k)}(S) \cup P_{s-1}(S)$. Assuming g_s has been defined on $P_s^{(k-1)}(S) \cup P_{s-1}(S)$ we extend over a k-simplex $\sigma \notin P_{s-1}(S)$ as follows:
Let $D = \operatorname{diam}(g_s(\partial\sigma))$ and choose $y \in g_s(\partial\sigma)$. By uniform contractibility of X relative to Y there exists D', depending only on D, and a null homotopy of $g_s|_{\partial\sigma}$ whose image lies inside $B_{D'}(y)$. We now extend g_s over σ using this null homotopy.
To construct a homotopy from $g_s \circ f_s$ to $i \colon Y \hookrightarrow X$ one does an induction over the simplices of $Y \times I$ (best using a cell structure such that all cells are of the form

$\sigma \times \{0\}, \sigma \times \{1\}$ or $\sigma \times I$) using the relative uniform contractibility of X with respect to Y. To do this boundedly one uses the uniform bound on the diameter of the cells of Y. $\square$

Remark 6.4. If $X = \coprod_{i\in I} X_i$ has FDC, is uniformly contractible with respect to a subcomplex $Y = \coprod_{i\in I} Y_i$ and there exists a subspace with bounded geometry $S = \coprod_{i\in I} S_i$ such that $Y \subseteq S^R$ for some R, then for any family $\mathcal{A}_i$ of additive categories the above theorem yields maps

$$(f_s)_*\colon \prod_{i\in I}^{bd}(\mathcal{A}_i)_c(Y_i) \to \prod_{i\in I}^{bd}(\mathcal{A}_i)_c(P_s(S_i)),\ (g_s)_*\colon \prod_{i\in I}^{bd}(\mathcal{A}_i)_c(P_s(S_i)) \to \prod_{i\in I}^{bd}(\mathcal{A}_i)_c(X_i)$$

and $(g_s)^* \circ (f_s)^*$ induces a map on K-theory that is homotopic to the inclusion $\prod_{i\in I}^{bd}(\mathcal{A}_i)_c(Y_i) \to \prod_{i\in I}^{bd}(\mathcal{A}_i)_c(X_i)$.

Theorem 6.5 ([RTY, Theorem 6.4])**.** *If $X = \coprod_{i\in I} X_i$ is a bounded geometry metric space with FDC, $\mathcal{A}$ an additive category, then for each $n \in \mathbb{Z}$ we have*

$$\operatorname*{colim}_s \pi_n \mathbb{K} \prod_{i\in I}^{bd} \mathcal{A}_c(P_s X_i) = \operatorname*{colim}_s \pi_n \mathbb{K}(\mathcal{A}_c(P_s X)) = 0.$$

The same proof as in [RTY] also yields a more general result.

Theorem 6.6. *If $X = \coprod_{i\in I} X_i$ is a bounded geometry metric space with FDC, $\{A_i\}_{i\in I}$ a family of additive categories, then for each $n \in \mathbb{Z}$ we have*

$$\operatorname*{colim}_s \pi_n \mathbb{K} \prod_{i\in I}^{bd} (\mathcal{A}_i)_c(P_s X_i) = 0.$$

7. The Descent Principle

Let Z be a simplicial Γ-CW complex and $\mathcal{A}$ a filtered, additive category with Γ-action. Let J_k be the set of k-simplices in the barycentric subdivision of Z. Since the vertices of every simplex in the barycentric subdivision are naturally ordered by the inclusion of the corresponding simplices in Z, we get maps

$$s_i\colon J_k \to J_{k-1}, \sigma \mapsto \partial_i \sigma \quad \text{for } 0 \le i \le k.$$

Define for each $n \in \mathbb{N}$

$$A_k^n := \operatorname{Map}_\Gamma(\Delta^k, (\mathbb{K} \prod_{J_k}^{bd} \mathcal{A})_n) \cong \operatorname{Map}(\Delta^k, (\mathbb{K} \prod_{J_k}^{bd} \mathcal{A})_n^\Gamma)$$

and

$$B_k^n := \prod_{i=0}^{k} \mathrm{Map}_\Gamma(\Delta^{k-1}, (\mathbb{K}\prod_{J_k}^{bd}\mathcal{A})_n),$$

where $(\mathbb{K}\prod_{J_k}^{bd}\mathcal{A})_n$ is the n-th space of the spectrum $\mathbb{K}\prod_{J_k}^{bd}\mathcal{A}$. The maps s_i induce maps $f_k^n := (s_i^*)_i \colon A_{k-1}^n \to B_k^n$ and the inclusions $d_i \colon \Delta^{k-1} \to \Delta^k$ induce maps $g_k^n := (d_i^*)_i \colon A_k^n \to B_k^n$.

Definiton 7.1. The *bounded mapping space* $\mathrm{Map}_\Gamma^{bd}(Z, \mathbb{K}\mathcal{A})$ is defined as the spectrum whose n-th space is the subspace of $\prod_{k\in\mathbb{N}} A_k^n$ consisting of all $(h_k)_k \in \prod_{k\in\mathbb{N}} A_k^n$ with $f_k^n(h_{k-1}) = g_k^n(h_k)$ for all $k \geq 1$. The structure maps are induced by the structure maps of the spectra $\mathbb{K}(\prod_{J_k}^{bd}\mathcal{A})$.

Remark 7.2. The inclusion $\prod_{J_k}^{bd}\mathcal{A} \hookrightarrow \prod_{J_k}\mathcal{A}$ induces a map

$$F_k \colon \mathrm{Map}_\Gamma(\Delta^k, \mathbb{K}\prod_{J_k}^{bd}\mathcal{A}) \to \mathrm{Map}_\Gamma(\Delta^k, \mathbb{K}\prod_{J_k}\mathcal{A}) \cong \mathrm{Map}_\Gamma(\coprod_{J_k}\Delta^k, \mathbb{K}\mathcal{A}).$$

Since $f_k^n(h_{k-1}) = g_k^n(h_k)$ for every $(h_k)_k \in \big(\mathrm{Map}_\Gamma^{bd}(Z, \mathbb{K}\mathcal{A})\big)_n$, the maps $F_k(h_k) \in \big(\mathrm{Map}_\Gamma(\coprod_{J_k}\Delta^k, \mathbb{K}\mathcal{A})\big)_n$ fit together to a map $h \in (\mathrm{Map}_\Gamma(Z, \mathbb{K}\mathcal{A}))_n$. Therefore, the inclusions $\prod_{J_k}^{bd}\mathcal{A} \hookrightarrow \prod_{J_K}\mathcal{A}$ induce a map

$$F \colon \mathrm{Map}_\Gamma^{bd}(Z, \mathbb{K}\mathcal{A}) \to \mathrm{Map}_\Gamma(Z, \mathbb{K}\mathcal{A}).$$

Furthermore, the diagonal map $\Delta \colon \mathcal{A} \to \prod_{J_k}^{bd}\mathcal{A}$ induces a map

$$\mathbb{K}(\mathcal{A})^\Gamma \to \mathrm{Map}_\Gamma^{bd}(Z, \mathbb{K}\mathcal{A})$$

by sending $x \in \mathbb{K}(\mathcal{A})_n^\Gamma$ to $(h_k)_k \in \mathrm{Map}_\Gamma^{bd}(Z, \mathbb{K}\mathcal{A})_n$ with $h_k \equiv \mathbb{K}(\Delta)(x)$ for all k.

Proposition 7.3. *Let Z be a simplicial Γ-CW complex and $\mathcal{A}$ a filtered, additive category with Γ-action. Let $(A_k^n, B_k^n, f_k^n, g_k^n)$ be as above. Then $\big(\mathrm{Map}_\Gamma^{bd}(Z, \mathcal{A})\big)_n$ is a model for the limit as well as the homotopy limit of the diagram $(A_k^n, B_k^n, f_k^n, g_k^n)$.*

We will use that pullbacks where one of the two maps is a fibration are homotopy pullbacks and that the limit of a tower of fibrations is a homotopy limit of that tower. These facts are well known and the analogous statements in the category of simplicial sets instead of topological spaces can be found in [BK72, Chapter XI, Examples 4.1(iv)&(v)].

Proof. Let $M_m \subseteq \prod_{k\leq m} A_k^n$ denote the subspace with $f_k^n(h_{k-1}) = g_k^n(h_k)$. M_m is a limit of the diagram $(A_k^n, B_k^n, f_k^n, g_k^n)_{k\leq m}$. The limit arises from taking finitely many pullbacks. Since the maps g_k^n are fibrations, the space M_m is a also a homotopy

limit of this diagram and the induced maps $M_m \to M_{m-1}$ are fibrations as well. $\left(\mathrm{Map}^{bd}_{\Gamma}(Z, \mathbb{K}\mathcal{A})\right)_n$ is a limit of the tower

$$\ldots \to M_m \to M_{m-1} \to \ldots \to M_1 \to M_0 = A_0,$$

and since all these arrows are fibrations, it is also a homotopy limit of the tower. Therefore, $\left(\mathrm{Map}^{bd}_{\Gamma}(Z, \mathbb{K}\mathcal{A})\right)_n$ is a model for the limit and the homotopy limit of $(A^n_k, B^n_k, f^n_k, g^n_k)$. □

Proposition 7.4. *Let Y be a finite dimensional, simplicial Γ-CW complex with finite stabilizers and let X be a Γ-CW complex such that for every Γ-set J with finite stabilizers*

$$\operatorname*{colim}_{K \subseteq X \text{ finite}} \pi_n \mathbb{K}\left(\prod_J^{bd} \mathbb{K}\mathcal{A}_\Gamma(\Gamma K))\right)^\Gamma = 0, \ \forall n \in \mathbb{N}.$$

Then

$$\operatorname*{colim}_{K \subseteq X \text{ finite}} \pi_n(\mathrm{Map}^{bd}_{\Gamma}(Y, \mathbb{K}\mathcal{A}_\Gamma(\Gamma K))) = 0, \ \forall n \in \mathbb{N}.$$

Proof. Let $x_0 \in S^n$ be the base point. As above let J_k be the set of k-simplices in the barycentric subdivision of Y and let $s_i \colon J_k \to J_{k-1}$ be defined by $\sigma \mapsto \partial_i \sigma$. An element in $\pi_n(\mathrm{Map}^{bd}_{\Gamma}(Y, \mathbb{K}\mathcal{A}_\Gamma(\Gamma K)))$ is represented by a system of maps

$$h_k \in \mathrm{Map}_*(S^n, \mathrm{Map}(\Delta^k, \mathbb{K}(\prod_{J_k}^{bd} \mathcal{A}_\Gamma(\Gamma K))^\Gamma)) \cong \mathrm{Map}_*(S^n \wedge \Delta^k_+, \mathbb{K}(\prod_{J_k}^{bd} \mathcal{A}_\Gamma(\Gamma K))^\Gamma)$$

such that:

- $h_k|_{S^n \wedge (\partial_i \Delta^k)_+} = (s_i)^* \circ h_{k-1}$

We will produce a null homotopy $\{H_k\}$ by induction on k. Since h_0 represents an element in $\pi_n \mathbb{K}(\prod_{J_0}^{bd} \mathcal{A}_\Gamma(\Gamma K))^\Gamma$, there exists $K \subseteq K'$ such that h_0 is null homotopic in $\mathbb{K}(\prod_{J_0}^{bd} \mathcal{A}_\Gamma(\Gamma K'))^\Gamma$ by assumption. Every such null homotopy gives a map

$$H_0 \in \mathrm{Map}_*(S^n \wedge \Delta^1, \mathbb{K}(\prod_{J_0}^{bd} \mathcal{A}_\Gamma(\Gamma K'))^\Gamma)$$

with

- $H_0|_{S^n \wedge (\partial_1 \Delta^1 \cup \{1\})} = h_0$.

where $\{1\} \in \Delta^1$ is the base point. Now assume we already have constructed maps

$$H_j \in \mathrm{Map}_*(S^n \wedge \Delta^{j+1}, \mathbb{K}(\prod_{J_j}^{bd} \mathcal{A}_\Gamma(\Gamma K'))^\Gamma)$$

($\{j+1\} \in \Delta^{j+1}$ being the base point) for $j < k$ such that

- $H_j|_{S^n\wedge\partial_i\Delta^{j+1}} = (s_i)^*\circ H_{j-1},\ 0\le i\le j$ and
- $H_j|_{S^n\wedge(\partial_{j+1}\Delta^{j+1}\cup\{j+1\})} = h_j$.

These glue together to a map

$$\tilde{H}_k \in \mathrm{Map}_*(S^n\wedge\partial\Delta^{k+1}, \mathbb{K}(\prod_{J_k}^{bd}\mathcal{A}_\Gamma(\Gamma K'))^\Gamma)$$

such that

- $\tilde{H}_k|_{S^n\wedge\partial_i\Delta^{k+1}} = (s_i)^*\circ H_{k-1},\ 0\le i\le k$ and
- $\tilde{H}_k|_{S^n\wedge(\partial_{k+1}\Delta^{k+1}\cup\{k+1\})} = h_k$.

Since

$$S^n\wedge\partial\Delta^{k+1}\cong S^{n+k}$$

the element $\tilde{H}_k$ gives an element in $\mathrm{Map}_*(S^{n+k}, \mathbb{K}(\prod_{J_k}^{bd}\mathcal{A}_\Gamma(\Gamma K'))^\Gamma)$. By assumption there exists $K'\subseteq K''$ such that $\tilde{H}_k$ is null homotopic in $\mathrm{Map}_*(S^{n+k}, \mathbb{K}(\prod_{J_k}^{bd}\mathcal{A}_\Gamma(\Gamma K''))^\Gamma)$. Any such null homotopic can be used to extend $\tilde{H}_k$ to a map

$$H_k \in \mathrm{Map}_*(S^n\wedge\Delta^{k+1}, \mathbb{K}(\prod_{J_k}^{bd}\mathcal{A}_\Gamma(\Gamma K''))^\Gamma)$$

with the properties stated above.

Since Y was assumed to be finite dimensional, after finitely many steps we have constructed the required null homotopy $\{H_k\}$. □

Lemma 7.5. *Let X be a metric space, then*

$$\prod_I^{bd}\mathcal{A}_\Gamma(X)_0 \to \prod_I^{bd}\mathcal{A}_\Gamma(X) \to \prod_I \mathcal{A}_\Gamma(X)^\infty$$

is a Karoubi filtration.

Proof. To show that the sequence is Karoubi it suffices to show that for any morphism $f\colon M\to N$ in $\mathcal{A}_\Gamma(X)^\infty$ and any $R>0$ there exists a morphism φ' in $\mathcal{A}_\Gamma(X)$ that is R-bounded and represents f.

Let φ be a representative of f. For every $x\in X$ let $U_x := B_{R/2}(x)\times[0,1]\subseteq X\times[0,1]$. Since φ is continuously controlled at 1, there exists a neighborhood $V_x\subseteq U_x$ of $(x,1)\in X\times[0,1]$ such that $\varphi_{(\gamma',y),(\gamma,v)}=0$ for all $\gamma,\gamma'\in\Gamma, v\in V_x, y\notin U_x$. Define $V := \bigcup_{x\in X}V_x$. Then $M|_{\Gamma\times X\times[0,1]\setminus\Gamma\times V}$ is an object in $\mathcal{A}_\Gamma(X)_0$ and therefore the morphism $\varphi'\colon M\to N$ defined by

$$\varphi'_{(\gamma',y),(\gamma,v)} = \begin{cases}\varphi_{(\gamma',y),(\gamma,v)} & v\in V\\ 0 & \text{else}\end{cases}$$

also represents f.
φ' is R-bounded, since $\varphi'_{(\gamma',y),(\gamma,v)} \neq 0$ implies $v \in V_x \subseteq B_{R/2}(x) \times [0,1]$ and $y \in U_x \subseteq B_{R/2}(x) \times [0,1]$ for some $x \in X$. Therefore, $d(\mathrm{pr}_X(v), \mathrm{pr}_X(y)) < R$, where $\mathrm{pr}_X \colon X \times [0,1] \to X$ is the projection. □

By formally defining $\prod_I^{bd} \mathcal{A}_\Gamma(X)^\infty := \prod_I \mathcal{A}_\Gamma(X)^\infty$ the above Proposition and Proposition 7.3 imply that we get the following homotopy fibration sequence:

$$\mathrm{Map}_\Gamma^{bd}(Z, \mathbb{K}(\mathcal{A}_\Gamma(X)_0)) \to \mathrm{Map}_\Gamma^{bd}(Z, \mathbb{K}(\mathcal{A}_\Gamma(X))) \to \mathrm{Map}_\Gamma^{bd}(Z, \mathbb{K}(\mathcal{A}_\Gamma(X)^\infty))$$

This can be used to prove the following version of the Descent Principle:

Theorem 7.6. *Let Γ be a discrete group admitting a finite dimensional model for $\underline{E}\Gamma$, let X be a Γ-CW complex, such that for every Γ-set J with finite stabilizers, every $K \subseteq X$ finite and every $x \in \pi_n(\mathbb{K}(\prod_{j\in J}^{bd} \mathcal{A}_\Gamma(\Gamma K))^\Gamma)$ there exists a finite subcomplex $K' \subseteq X$ containing K such that under the map induced by the inclusion*

$$\prod_{j\in J}^{bd} \mathcal{A}_\Gamma(\Gamma K) \to \prod_{j\in J}^{bd} \mathcal{A}_\Gamma(\Gamma K')$$

x maps to zero. Then the map

$$H_*^\Gamma(X; \mathbb{K}_\mathcal{A}) \to K_*(\mathcal{A}[\Gamma])$$

is a split injection.

Proof. Let Y be a finite dimensional, simplicial Γ-CW-model for $\underline{E}\Gamma$.
Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{K}(\mathcal{A}_\Gamma(\Gamma K)_0)^\Gamma \arrow[r] \arrow[d] & \mathbb{K}(\mathcal{A}_\Gamma(\Gamma K))^\Gamma \arrow[r] \arrow[d] & \mathbb{K}(\mathcal{A}_\Gamma(\Gamma K)^\infty)^\Gamma \arrow[d, "f"] \\
\mathrm{Map}_\Gamma^{bd}(Y, \mathbb{K}(\mathcal{A}_\Gamma(\Gamma K)_0)) \arrow[r] \arrow[d] & \mathrm{Map}_\Gamma^{bd}(Y, \mathbb{K}(\mathcal{A}_\Gamma(\Gamma K))) \arrow[r] \arrow[d] & \mathrm{Map}_\Gamma^{bd}(Y, \mathbb{K}(\mathcal{A}_\Gamma(\Gamma K)^\infty)) \arrow[d, "g"] \\
\mathrm{Map}_\Gamma(Y, \mathbb{K}(\mathcal{A}_\Gamma(\Gamma K)_0)) \arrow[r] & \mathrm{Map}_\Gamma(Y, \mathbb{K}(\mathcal{A}_\Gamma(\Gamma K))) \arrow[r] & \mathrm{Map}_\Gamma(Y, \mathbb{K}(\mathcal{A}_\Gamma(\Gamma K)^\infty))
\end{tikzcd}$$

All three rows in this diagram are induced by Karoubi filtrations and are, therefore, homotopy fibrations. The vertical rows are those from Remark 7.2. The composition $g \circ f$ is a weak homotopy equivalence by [Ros04, Theorem 6.2]. Therefore, f induces a split injection on homotopy groups.
(Since K-theory commutes with products, g is a weak homotopy equivalence but we do not need this fact here.)

Since $\operatorname{colim}_{K\subseteq X \text{ finite}} \pi_n(\mathbb{K}(\mathcal{A}_\Gamma(\Gamma K))) \cong \pi_n\mathbb{K}(\mathcal{A}_{p,\Gamma}(X))$, for $p\colon X \to \Gamma\backslash X$ the projection map, after taking homotopy groups and colimits over $K \subseteq X$ finite we get the following:

$$\begin{tikzcd}
\pi_{n+1}\mathbb{K}(\mathcal{A}_{p,\Gamma}(X)^\infty)^\Gamma \arrow[r, "\partial"] \arrow[d, "f_*"] & \pi_n\mathbb{K}(\mathcal{A}_{p,\Gamma}(X)_0) \arrow[d] \\
\operatorname{colim}_K \pi_{n+1}(\operatorname{Map}^{bd}_\Gamma(Y, \mathbb{K}(\mathcal{A}_\Gamma(\Gamma K)^\infty))) \arrow[r, "\partial"] & \operatorname{colim}_K \pi_n(\operatorname{Map}^{bd}_\Gamma(Y, \mathbb{K}(\mathcal{A}_\Gamma(\Gamma K)_0)))
\end{tikzcd}$$

The lower horizontal map is an isomorphism by Proposition 7.4 and f_* is split injective as stated above. So the upper horizontal map is split injective. This map is equivalent to the map in the theorem. □

8. Proof of the main theorem

Theorem 8.1. *Let Γ be a discrete group with fqFDC and with an upper bound on the order of its finite subgroups and let $\mathcal{A}$ be a small additive Γ-category. Assume that there is a finite dimensional Γ-CW-model for the universal space for proper Γ-actions $\underline{E}\Gamma$.*
Then the assembly map in algebraic K-theory $H^\Gamma_*(\underline{E}\Gamma; \mathbb{K}_\mathcal{A}) \to K_*(\mathcal{A}[\Gamma])$ is a split injection.

Let X be a finite dimensional, simplicial Γ-CW-model for $\underline{E}\Gamma$ with a proper Γ-invariant metric. Such a model exists by the assumptions of Theorem 8.1 and Proposition 1.5.
Let G be a finite subgroup of Γ. Let $G = H^G_0, H^G_1, \ldots, H^G_{m_G} = \{e\}$ be a representing system for the conjugacy classes of subgroups of G ordered by cardinality, that is $|H^G_i| \geq |H^G_{i+1}|$.
Let $m := \max_G m_G$ and define $H^G_l := \{e\}$ for all $m_G \leq l \leq m$.

For each $k,\ 0 \leq k \leq m$, define $\mathcal{S}^G_k := \{(H^G_i)^g \mid 0 \leq i \leq k, g \in G\}$ and $Z^G_k := X^{\mathcal{S}^G_k}$. Since $\mathcal{S}^G_k$ is closed under conjugation by G, the space Z^G_k is G-invariant for every k.

Lemma 8.2. *Under the assumptions of Theorem 8.1 for every k, $0 \leq k \leq m$, every n and every family of finite subgroups $\{G_i\}_{i\in I}$ of Γ the following holds*

$$\operatorname*{colim}_K \pi_n\mathbb{K}\left(\prod^{bd}_{i\in I} \mathcal{A}[H^{G_i}_k]_c(G_i\backslash(Z^{G_i}_k \cap \Gamma K))\right) = 0,$$

where the colimit is taken over all compact subsets $K \subseteq X$.

Proof. Let $K \subseteq X$ be compact. By Lemma 1.8 and Lemma 1.10 the space $\coprod_{i\in I} G_i\backslash Z^{G_i}_k$ is uniformly contractible with respect to $\coprod_{i\in I} G_i\backslash(Z^{G_i}_k \cap \Gamma K)$.

Let $x \in K$ and let $R > 0$ such that $K \subseteq B_R(x)$. Since $Z_k^{G_i} \cap \Gamma K$ is G_i-invariant, $(Z_k^{G_i} \cap \Gamma K)^R$ is G_i-invariant as well and $G_i \backslash (Z_k^{G_i} \cap \Gamma K) \subseteq G_i \backslash (\Gamma x \cap (Z_k^{G_i} \cap \Gamma K)^R)^R$. Choose maps $\rho_i \colon G_i \backslash (\Gamma x \cap (Z_k^{G_i} \cap \Gamma K)^R) \to G_i \backslash (Z_k^{G_i} \cap \Gamma K)$ with $d(y, \rho_i(y)) \leq R$ for all $y \in G_i \backslash (\Gamma x \cap (Z_k^{G_i} \cap \Gamma K)^R)$ and define $S_i := \mathrm{im}(\rho_i)$.
We have $G_i \backslash (\Gamma K \cap Z_k^{G_i}) \subseteq S_i^{2R}$ and $\coprod_{i \in I} S_i \subseteq \coprod_{i \in I} G_i \backslash (Z_k^{G_i} \cap \Gamma K)$ is a subspace with bounded geometry because Γx has bounded geometry. So by Proposition 6.3 and Remark 6.4 for every $s > 4R$ there exist $K' \subseteq X$ compact and containing K such that the inclusion

$$\mathbb{K}\left(\prod_{i \in I}^{bd} \mathcal{A}[H_k^{G_i}]_c(G_i \backslash (Z_k^{G_i} \cap \Gamma K))\right) \to \mathbb{K}\left(\prod_{i \in I}^{bd} \mathcal{A}[H_k^{G_i}]_c(G_i \backslash (Z_k^{G_i} \cap \Gamma K'))\right)$$

factorizes over $\mathbb{K}\left(\prod_{i \in I}^{bd} \mathcal{A}[H_k^{G_i}]_c(P_s(S_i))\right)$.
Define a metric on Γ by

$$d(\gamma, \gamma') = \begin{cases} 0 & \gamma = \gamma' \\ 2 + d(\gamma x, \gamma' x) & \gamma \neq \gamma' \end{cases}.$$

Since the stabilizer of x is finite, this metric is proper. The map $\Gamma \to \Gamma x, \gamma \mapsto \gamma x$ is a coarse equivalence and therefore Γx has fqFDC.
$\coprod_{i \in I} S_i$ is coarsely equivalent to a subspace of $\coprod_{i \in I} G_i \backslash \Gamma x$ and so $\coprod_{i \in I} S_i$ has FDC. It follows that $\mathrm{colim}_s \, \pi_n \mathbb{K}\left(\prod_{i \in I}^{bd} \mathcal{A}[H_k^{G_i}]_c(P_s(S_i))\right) = 0$ by Theorem 6.6. Therefore, also the colimit in the statement of the proposition vanishes. □

The following lemma is essentially the same as [BR07b, 8.4].

Lemma 8.3. *For each k, $1 \leq k \leq m$, every finite subgroup $G \leq \Gamma$ and every compact subset $K \subseteq X$ we have the following isomorphism:*

$$\mathcal{A}_\Gamma^G(Z_k^G \cap \Gamma K, Z_{k-1}^G \cap \Gamma K) \cong \mathcal{A}[H_k^G]_c(G \backslash (Z_k^G \cap \Gamma K), G \backslash (Z_{k-1}^G \cap \Gamma K)).$$

Proof of Theorem 8.1
By the Descent Principle it suffices to show that for every integer n and every Γ-set J with finite stabilizers the following holds

$$\operatorname*{colim}_K \pi_n \mathbb{K}\left(\prod_{j \in J}^{bd} \mathcal{A}_\Gamma(\Gamma K)\right)^\Gamma = 0$$

where the colimit is taken over all compact subspaces $K \subseteq X$.
But since $\left(\prod_{j\in J}^{bd} \mathcal{A}_\Gamma(\Gamma K)\right)^\Gamma$ is equivalent to $\prod_{\Gamma j\in\Gamma\backslash J}^{bd} \mathcal{A}_\Gamma^{\Gamma_j}(\Gamma K)$, where Γ_j is the stabilizer of $j \in J$, this is equivalent to showing that for every family of finite subgroups $\{G_i\}_{i\in I}$ over some index set I the following holds

$$\operatorname*{colim}_K \pi_n \mathbb{K}\left(\prod_{i\in I}^{bd} \mathcal{A}_\Gamma^{G_i}(\Gamma K))\right) = 0.$$

We will proceed by induction on the filtrations

$$\underline{\mathrm{E}}\Gamma^{G_i} = Z_0^{G_i} \subseteq Z_1^{G_i} \subseteq \ldots \subseteq Z_{m-1}^{G_i} \subseteq Z_m^{G_i} = \underline{\mathrm{E}}\Gamma$$

defined above.
Since G_i acts trivially on $\underline{\mathrm{E}}\Gamma^{G_i}$, $\mathcal{A}_\Gamma^{G_i}(\underline{\mathrm{E}}\Gamma^{G_i} \cap \Gamma K)$ is equivalent to $\mathcal{A}[G_i]_c(\underline{\mathrm{E}}\Gamma^{G_i} \cap \Gamma K)$ and by Lemma 8.2 we have

$$\operatorname*{colim}_K \pi_n \mathbb{K} \prod_{i\in I}^{bd} \mathcal{A}[G_i]_c(\underline{\mathrm{E}}\Gamma^{G_i} \cap \Gamma K) = 0.$$

This completes the base case of the induction.

Assume now

$$\operatorname*{colim}_K \pi_n \mathbb{K} \prod_{i\in I}^{bd} \mathcal{A}_\Gamma^{G_i}(Z_k^{G_i} \cap \Gamma K) = 0.$$

We must show that also

$$\operatorname*{colim}_K \pi_n \mathbb{K} \prod_{i\in I}^{bd} \mathcal{A}_\Gamma^{G_i}(Z_{k+1}^{G_i} \cap \Gamma K) = 0.$$

Consider the Karoubi filtration

$$\prod_{i\in I}^{bd} \mathcal{A}_\Gamma^{G_i}(Z_k^{G_i} \cap \Gamma K) \to \prod_{i\in I}^{bd} \mathcal{A}_\Gamma^{G_i}(Z_{k+1}^{G_i} \cap \Gamma K) \to \prod_{i\in I}^{bd} \mathcal{A}_\Gamma^{G_i}(Z_{k+1}^{G_i} \cap \Gamma K, Z_k^{G_i} \cap \Gamma K))$$

which yields a homotopy fibration of spectra after applying $\mathbb{K}$. By using the induction hypothesis, we only need to show that

$$\operatorname*{colim}_K \pi_n \mathbb{K} \prod_{i\in I}^{bd} \mathcal{A}_\Gamma^{G_i}(Z_{k+1}^{G_i} \cap \Gamma K, Z_k^{G_i} \cap \Gamma K) = 0.$$

By Lemma 8.3

$$\prod_{i\in I}^{bd} \mathcal{A}_\Gamma^{G_i}(Z_{k+1}^{G_i} \cap \Gamma K, Z_k^{G_i} \cap \Gamma K)$$

is equivalent to

$$\prod_{i\in I}^{bd} \mathcal{A}[H_k^{G_i}]_c(G_i\backslash(Z_{k+1}^{G_i}\cap \Gamma K), G_i\backslash(Z_k^{G_i}\cap \Gamma K)),$$

which fits into the Karoubi sequence

$$\prod_{i\in I}^{bd} \mathcal{A}[H_k^{G_i}]_c(G_i\backslash(Z_k^{G_i}\cap \Gamma K)) \to \prod_{i\in I}^{bd} \mathcal{A}[H_k^{G_i}]_c(G_i\backslash(Z_{k+1}^{G_i}\cap \Gamma K))$$

$$\to \prod_{i\in I}^{bd} \mathcal{A}[H_k^{G_i}]_c(G_i\backslash(Z_{k+1}^{G_i}\cap \Gamma K), G_i\backslash(Z_k^{G_i}\cap \Gamma K)).$$

By Lemma 8.2 we have

$$\operatorname*{colim}_K \pi_n \mathbb{K} \prod_{i\in I}^{bd} \mathcal{A}[H_k^{G_i}]_c(G_i\backslash(Z_k^{G_i}\cap \Gamma K)) = \operatorname*{colim}_K \pi_n \mathbb{K} \prod_{i\in I}^{bd} \mathcal{A}[H_k^{G_i}]_c(G_i\backslash(Z_{k+1}^{G_i}\cap \Gamma K)) = 0.$$

Therefore, also

$$\operatorname*{colim}_K \pi_n \mathbb{K} \prod_{i\in I}^{bd} \mathcal{A}[H_k^{G_i}]_c(G_i\backslash(Z_{k+1}^{G_i}\cap \Gamma K), G_i\backslash(Z_k^{G_i}\cap \Gamma K)) = 0.$$

□

9. L-theory

As in [BR07b] we get the following L-theoretic version of Theorem 8.1.

Theorem 9.1. *Let Γ be a discrete group with fqFDC and $\mathcal{A}$ a small additive Γ-category with involution. Assume that there is a finite dimensional Γ-CW-model for the universal space for proper Γ-actions $\underline{E}\Gamma$ and that there is an upper bound on the order of finite subgroups of Γ. Assume further that for every finite subgroup $G \leq \Gamma$ there is an $i_0 \in \mathbb{N}$ such that for $i \geq i_0$, $K_{-i}(\mathcal{A}[G]) = 0$, where $\mathcal{A}$ is considered only as an additive category.*
Then the assembly map in L-theory $H_*^\Gamma(\underline{E}\Gamma; \mathbb{L}_\mathcal{A}^{\langle -\infty\rangle}) \to L_*^{\langle -\infty\rangle}(\mathcal{A}[\Gamma])$ is a split injection.

Proof. Everything we did works for L-theory as it works for K-theory with exception of the Descent Principle 7.6. Here the additional assumption about the vanishing of $K_{-i}(\mathcal{A}[G])$ for large i is needed because only then the L-theoretic analogue of [Ros04, Theorem 6.3] holds. For more details on this see [BR07b] and [CP95, Section 4]. □

References

[ACFP94] Douglas R. Anderson, Francis X. Connolly, Steven C. Ferry, and Erik K. Pedersen, *Algebraic K-theory with continuous control at infinity*, J. Pure Appl. Algebra **94** (1994), no. 1, 25–47. MR 1277522 (95b:19003)

[AS81] Roger C. Alperin and Peter B. Shalen, *Linear groups of finite cohomological dimension*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), no. 3, 339–341. MR 609046 (82c:20087)

[BK72] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin, 1972. MR 0365573 (51 #1825)

[BR07a] Arthur Bartels and Holger Reich, *Coefficients for the Farrell-Jones conjecture*, Adv. Math. **209** (2007), no. 1, 337–362. MR 2294225 (2008a:19002)

[BR07b] Arthur Bartels and David Rosenthal, *On the K-theory of groups with finite asymptotic dimension*, J. Reine Angew. Math. **612** (2007), 35–57. MR 2364073 (2009a:19004)

[CH69] E. H. Connell and John Hollingsworth, *Geometric groups and Whitehead torsion*, Trans. Amer. Math. Soc. **140** (1969), 161–181. MR 0242151 (39 #3485)

[CP95] Gunnar Carlsson and Erik Kjær Pedersen, *Controlled algebra and the Novikov conjectures for K- and L-theory*, Topology **34** (1995), no. 3, 731–758. MR 1341817 (96f:19006)

[DP] Dieter Degrijse and Nansen Petrosyan, *Bredon cohomological dimensions for groups acting on CAT(0)-spaces*, arXiv:1208.3884.

[Gro93] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR 1253544 (95m:20041)

[GTYa] Erik Guentner, Romain Tessera, and Guoliang Yu, *Discrete groups with finite decompositon complexity*, to appear in Groups, Geometry and Dynamics.

[GTYb] ———, *A notion of geometric complexity and its application to topological rigidity*, Inventiones Mathematicae, 1–43, 10.1007/s00222-011-0366-z.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354 (2002k:55001)

[Mis01] Guido Mislin, *On the classifying space for proper actions*, Cohomological methods in homotopy theory (Bellaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 263–269. MR 1851258 (2002f:55032)

[PW85] Erik K. Pedersen and Charles A. Weibel, *A nonconnective delooping of algebraic K-theory*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 166–181. MR 802790 (87b:18012)

[Qui79] Frank Quinn, *Ends of maps. I*, Ann. of Math. (2) **110** (1979), no. 2, 275–331. MR 549490 (82k:57009)

[Qui82] ———, *Ends of maps. II*, Invent. Math. **68** (1982), no. 3, 353–424. MR 669423 (84j:57011)

[Roe03] John Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, American Mathematical Society, Providence, RI, 2003. MR 2007488 (2004g:53050)

[Ros04] David Rosenthal, *Splitting with continuous control in algebraic K-theory*, K-Theory **32** (2004), no. 2, 139–166. MR 2083578 (2005g:19003)

[RTY] Daniel A. Ramras, Romain Tessera, and Guoliang Yu, *Finite decomposition complexity and the integral Novikov conjecture for higher algebraic K-theory*, to appear in Crelle's Journal (J. reine angew. Math.)

[Sel60] Atle Selberg, *On discontinuous groups in higher-dimensional symmetric spaces*, Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960), Tata Institute of Fundamental Research, Bombay, 1960, pp. 147–164. MR 0130324 (24 #A188)

Fachbereich Mathematik und Informatik, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany
E-mail address: d_kasp01@math.uni-muenster.de